

Ques 1:- State and prove Cauchy's root test theorem.

Let  $u_n$  be always positive and  $\lim_{n \rightarrow \infty} u_n^{1/n} = l$

then (i) If  $l < 1$ ,  $\sum u_n$  is convergent

(ii) If  $l > 1$ ,  $\sum u_n$  is divergent

(iii) If  $l = 1$ , the test fails and the series may be either convergent or divergent.

Proof:- let  $l < 1$

Since  $l < 1$ , we can choose  $\epsilon > 0$  such that  $l + \epsilon < 1$

let  $l + \epsilon = r$  so that  $0 < r < 1$

Since  $\lim_{n \rightarrow \infty} u_n^{1/n} = l$  therefore there exists positive integer  $m$  such that

$$|u_n^{1/n} - l| < \epsilon \text{ when } n \geq m.$$

$$\text{i.e. } 1 - \epsilon < u_n^{1/n} < 1 + \epsilon \text{ when } n \geq m$$

$$\text{i.e. } (1 - \epsilon)^n < u_n < (1 + \epsilon)^n \text{ when } n \geq m$$

Taking the last inequality we find that

$$u_n < r^n \text{ where } r = 1 + \epsilon < 1 \text{ for all } n \geq m.$$

But the series  $r^m + r^{m+1} + \dots$  is a series in G.P. whose common ratio  $r < 1$  and hence it is convergent. Therefore by the comparison test

$$u_m + u_{m+1} + \dots \text{ is convergent}$$

Hence  $\sum u_n$  is convergent

(ii) let  $l > 1$

Since  $l > 1$ , we can choose  $\epsilon > 0$  such that

$$l - \epsilon > 1$$

let  $l - \epsilon = R$  so that  $R > 1$

Since  $\lim_{n \rightarrow \infty} u_n^{y_n} = 1$  therefore there exists a positive integer  $m$  (2)  
 such that

$$|u_n^{y_n} - 1| < \epsilon \text{ when } n \geq m$$

i.e.  $1 - \epsilon < u_n^{y_n} < 1 + \epsilon$  when  $n \geq m$

i.e.  $(1 - \epsilon)^n < u_n < (1 + \epsilon)^n$  when  $n \geq m$

Taking the first inequality, we find that

$$u_n > R^n \text{ where } R = 1 - \epsilon > 1 \text{ for all } n \geq m.$$

Thus we find that  $u_n$  does not tend to zero. Hence  $\sum u_n$  is not convergent. But a series of positive terms must either converge or diverge and so  $\sum u_n$  is divergent.

Case III. We will illustrate this case when  $l=1$  by taking suitable examples.

Let  $u_n = \frac{1}{n}$  then  $u_n^{y_n} = \left(\frac{1}{n}\right)^{y_n} = \frac{1}{n^{y_n}}$

$$\therefore \lim_{n \rightarrow \infty} u_n^{y_n} = \frac{1}{\lim_{n \rightarrow \infty} n^{y_n}} = \frac{1}{1} = 1$$

But  $\sum \frac{1}{n}$  is a divergent series

Thus we see in this case that when  $\lim_{n \rightarrow \infty} u_n^{y_n} = 1$  the series  $\sum u_n$  is divergent

on the other hand let  $u_n = \frac{1}{n^2}$   
 then in this case also  $u_n^{y_n} = \frac{1}{n^{2y_n}} = \frac{1}{(n^{y_n})^2}$

$$\therefore \lim_{n \rightarrow \infty} u_n^{y_n} = 1$$

But the series  $\sum \frac{1}{n^2}$  is a convergent series

Thus we see that in this case when  $\lim_{n \rightarrow \infty} u_n^{y_n} = 1$  the series  $\sum u_n$  is convergent.

These two examples show that if  $\lim_{n \rightarrow \infty} u_n^{y_n} = 1$  the series  $\sum u_n$  may be either convergent or divergent that is the test fails